Georgia State University
ScholarWorks @ Georgia State University

Mathematics Theses

Department of Mathematics and Statistics

5-3-2007

Spectrally Arbitrary and Inertially Arbitrary Sign Pattern Matrices

Nilay Sezin Demir

SPECTRALLY ARBITRARY AND INERTIALLY ARBITRARY SIGN PATTERN MATRICES

by

NILAY SEZIN DEMIR

Under the Direction of Dr. Frank J. Hall

Abstract

A sign pattern matrix (or a sign pattern, or a pattern) is a matrix whose entries are from the set $\{+, -, 0\}$. An $n \times n$ sign pattern matrix is a spectrally arbitrary pattern (SAP) if for every monic real polynomial $p(x)$ of degree n, there exists a real matrix B whose entries agree in sign with A such that the characteristic polynomial of B is $p(x)$. An $n \times n$ sign pattern A is an inertially arbitrary pattern (IAP) if (r, s, t) belongs to the inertia set of A for every nonnegative integer triple (r, s, t) with $r + s + t = n$. Some elementary results on these two classes of sign patterns are first exhibited. Tree sign patterns are then investigated, with a special emphasis on 4×4 tridiagonal sign patterns. Connections between the SAP (IAP) classes and the classes of potentially nilpotent and potentially stable patterns are explored. Some interesting open questions are also provided.

Keywords: sign pattern matrix; spectrally arbitrary pattern; inertially arbitrary pattern; tree sign pattern; potentially nilpotent pattern; potentially stable pattern; Gröbner basis

SPECTRALLY ARBITRARY AND INERTIALLY ARBITRARY SIGN PATTERN MATRICES

by

NILAY SEZIN DEMIR

A Thesis Presented in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in College of Arts and Sciences

Georgia State University

2007

SPECTRALLY ARBITRARY AND INERTIALLY ARBITRARY SIGN PATTERN MATRICES

by

NILAY SEZIN DEMIR

Major Professor:	Frank Hall
Committee:	Zhongshan Li
	Marina Arav

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

April 2007

Acknowledgements

The author wishes to gratefully acknowledge the assistance of Dr. Frank J. Hall and Dr. Zhongshan Li without whose guidance this thesis would not have been possible.

TABLE OF CONTENTS

Acknowledgements iv

1. Introduction and Preliminaries 1

2. Some Elementary Results 6

3. Tree Sign Patterns 15

4. 4×4 Tridiagonal Spectrally Arbitrary Sign Patterns 19

5. Connections with Potentially Nilpotent Patterns 26

6. Some Open Questions 30

References 31

1. INTRODUCTION AND PRELIMINARIES

The origins of sign pattern matrices are in the book [Sam47] by the Nobel Economics Prize winner P. Samuelson, who pointed to the need to solve certain problems in economics and other areas based only on the signs of the entries of the matrices. The study of sign pattern matrices has become somewhat synonymous with qualitative matrix analysis. The dissertation of C. Eschenbach [Esc87], directed by C.R. Johnson, studied sign pattern matrices that "require" or "allow" certain properties and summarized the work on sign patterns up to that point. In 1995, Richard Brualdi and Bryan Shader produced a thorough treatment [BS95] on sign pattern matrices from the sign-solvability vantage point. Since 1995 there has been a considerable number of papers on sign patterns. For a current survey with extensive bibliography, see Hall and Li [HL07]. We further note that because of the interplay between sign pattern matrices and graph theory, the study of sign patterns is regarded as a part of combinatorial matrix theory.

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern, pattern). We remark that in this thesis we mostly use $\{+, -, 0\}$ notation for sign patterns, whereas in the literature $\{1, -1, 0\}$ notation is also commonly used, such as in [BS95]. We denote the set of all $n \times n$ sign pattern matrices by Q_n. For a real matrix B, $sgn(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the *sign pattern class of* A is defined by

$$Q(A) = \{\, B \,:\, \mathrm{sgn}(B) = A \,\}.$$

A *subpattern* of a sign pattern A is a sign pattern matrix obtained from A by replacing a number (possibly none) of the $+$ or $-$ entries in A with 0. If $\hat{A}$ is a subpattern of A, we also say that A is a *superpattern* of $\hat{A}$.

The sign pattern $I_n \in Q_n$ is the diagonal pattern of order n with $+$ diagonal entries. A sign pattern matrix P is called a *permutation pattern* if exactly one entry in each row and column is equal to $+$, and all other entries are 0. Two sign pattern matrices A_1 and A_2 are said to be *permutationally equivalent* if there are permutation patterns P_1 and P_2 such that $A_2 = P_1 A_1 P_2$; they are said to be *permutationally similar* if there is a permutation pattern P such that $A_2 = P^T A_1 P$.

A *signature (sign) pattern* is a diagonal sign pattern all of whose diagonal entries are nonzero. Two sign pattern matrices A_1 and A_2 are said to be *signature equivalent* if there are signature patterns S_1 and S_2 such that $A_2 = S_1 A_1 S_2$, and more specifically *signature similar* if there is a signature pattern S such that $A_2 = S A_1 S$.

A sign pattern $A \in Q_n$ is said to be *sign nonsingular* if every matrix $B \in Q(A)$ is nonsingular. It is well known that A is sign nonsingular if and only if $\det A = +$ or $\det A = -$, that is, in the standard expansion of $\det A$ into $n!$ terms, there is at least one nonzero term, and all the nonzero terms have the same sign. This means that $\det B$ is positive (or negative) for all $B \in Q(A)$. A is said to be *sign singular* if every matrix $B \in Q(A)$ is singular, or equivalently, if $\det A = 0$.

A *combinatorially symmetric sign pattern matrix* is a square sign pattern A where $a_{ij} \neq 0$ iff $a_{ji} \neq 0$. The *graph G(A) of a combinatorially symmetric* $n \times n$ sign pattern matrix $A = [a_{ij}]$ is the graph with vertex set $\{1, 2, 3, \ldots, n\}$ where $\{i, j\}$ is an edge iff $a_{ij} \neq 0$. A *tree sign pattern* (tsp) matrix is a combinatorially symmetric sign pattern matrix whose graph is a tree (possibly with loops).

If $A = [a_{ij}]$ is an $n \times n$ sign pattern matrix, then a *(simple) cycle* of *length* k (or a *k-cycle*) in A is a formal product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}$, where each of the elements is nonzero and the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices. The *sign* (positive or negative) of a simple cycle in a sign pattern A is the actual product of the entries in the cycle, following the obvious rules that multiplication is commutative and associative, and $(+)(+) = +$, $(+)(-) = -$.

A *composite cycle* γ in A is a product of simple cycles, say $\gamma = \gamma_1\gamma_2\ldots\gamma_m$, where the index sets of the γ_i's are mutually disjoint. If the length of γ_i is l_i, then the length of γ is $\sum_{i=1}^{m} l_i$, and the *signature* of γ is $(-)^{\sum_{i=1}^{m}(l_i-1)}$. A cycle γ is *odd* (*even*) when the length of the simple or composite cycle γ is odd (even). We note also that if A is an $n \times n$ sign pattern, then each nonzero term in $\det A$ is equal to the product of the signature and sign of a cycle of length n.

The set of all eigenvalues (counting multiplicities) of a square matrix B is denoted by $\sigma(B)$, and the *inertia of matrix* B is the ordered triple

$$i(B) = (i_+(B), i_-(B), i_0(B)),$$

in which $i_+(B)$, $i_-(B)$ and $i_0(B)$ are the numbers of elements of $\sigma(B)$ with positive, negative and zero real parts, respectively. The *inertia set* of a square sign pattern A is the set of ordered triples $i(A) = \{i(B) \ : \ B \in Q(A)\ \}$. An $n \times n$ sign pattern A is said to be an *inertially arbitrary pattern* (IAP) if $(r, s, t) \in i(A)$ for every nonnegative integer triple (r, s, t) with $r + s + t = n$.

An $n \times n$ matrix B is *stable* if $i(B) = (0, n, 0)$. An $n \times n$ pattern A is *sign stable* if $i(A) = \{(0, n, 0)\}$, and *potentially stable* if $(0, n, 0) \in i(A)$.

An $n \times n$ pattern A is a *spectrally arbitrary pattern* (SAP) if, for any given real monic polynomial $r(x)$ of degree n, there is a matrix $B \in Q(A)$ with characteristic polynomial $r(x)$. That is, A is an SAP if there exists $B \in Q(A)$ having any possible spectrum of a real matrix, namely any set of n complex numbers with any nonreals occuring as conjugate pairs. Clearly, if A is an SAP, then A is an IAP. We give examples to show that the converse does not hold in Chapter 5.

Every SAP pattern A must allow nilpotence (that is to say, A is *potentially nilpotent*). This follows by using $r(x) = x^n$. Also, by using $r(x) = (x + 1)^n$, we observe that every SAP pattern A is potentially stable. In fact, every IAP is potentially stable, since in particular, $(0, n, 0) \in i(A)$.

Of course, not every potentially nilpotent pattern is even an IAP pattern.

$$\begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix}$$

is an example of a potentially nilpotent pattern that is not an IAP.

$$\begin{bmatrix} - & + \\ 0 & - \end{bmatrix}$$

is an example of a potentially stable pattern that is not an IAP.

A sign pattern A is a *minimal inertially arbitrary pattern* (MIAP), if A is an IAP, but is not an IAP if one or more nonzero entries is replaced by zero.

Analogously, A is a *minimal spectrally arbitrary pattern* (MSAP), if A is an SAP, but is not an SAP if one or more nonzero entries is replaced by zero.

We make the following useful general observation. If we know that a property holds for every zero/nonzero pattern of order n, then we know that the property holds in particular, for every sign pattern of order n.

There has been considerable interest recently in spectrally arbitrary sign patterns which were introduced in [DJO00]. In [BMOD04] it was established that any spectrally arbitrary sign pattern of order n must have at least $2n - 1$ nonzero entries and conjectured that any spectrally arbitrary sign pattern of order n must have at least $2n$ nonzero entries. (This is known as the *$2n$-conjecture*). In [BMOD04] and also in [CV05] the 3×3 spectrally arbitrary sign patterns were classified and demonstrated to have at least six nonzero entries.

Spectrally arbitrary tree sign patterns, especially those whose graph (excluding loops) is a path, are considered in [DJO00]. A method, based on the implicit function theorem, for proving that a pattern (and all superpatterns) is an SAP is developed there. A full class of spectrally arbitrary patterns is constructed in

[MOT03] by using a Soules matrix. The implicit function theorem method is used in [BMOD04] to show that some Hessenberg sign patterns are minimal SAPs, the first such families for all orders to be presented. Other spectrally arbitrary sign pattern classes are constructed in [CV05] also by using the implicit function theorem method. All potentially stable star sign patterns are characterized in [GL01]. The inertias of matrices having a symmetric star sign pattern are characterized in [SSG04]. Potentially nilpotent star sign patterns are considered in [Yeh96], in which explicit characterization are given for patterns of orders two and three, and a recursive characterization for patterns of general order n is proved.

2. SOME ELEMENTARY RESULTS

We begin with the following foundational result, the proof of which is clear.

Lemma 2.1. The class of all $n \times n$ SAP's (IAP's) is closed under negation, transposition, permutational similarity, and signature similarity.

Recall that for an $n \times n$ matrix B, the characteristic polynomial of B is

$$p_B(x) = x^n - E_1(B)x^{n-1} + E_2(B)x^{n-2} - \ldots + (-1)^n E_n(B),$$

where $E_k(B)$ is the sum of all the $k \times k$ principal minors of B. Note that $E_k(B)$ is also equal to the symmetric sum $S_k(\lambda_1, \ldots, \lambda_n)$, the sum of all k-fold products of distinct items from $\sigma(A) = \{\lambda_1, \lambda_2, \ldots \lambda_n\}$. More precisely,

$$S_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}.$$

Theorem 2.2. If A is an $n \times n$ SAP, then $Q(A)$ allows a positive and a negative principal minor of order k for all $k = 1, 2, \ldots, n$.

Proof. Since A is an SAP, there exists $B \in Q(A)$ such that $p_B(x) = x^n + x^{n-k}$, so that $E_k(B) = 1$ for k even, and $E_k(B) = -1$ for k odd. Similarly, since A is an SAP, there exists $C \in Q(A)$ such that $p_C(x) = x^n - x^{n-k}$, so that $E_k(C) = -1$ for k even, and $E_k(C) = 1$ for k odd. Hence, some $k \times k$ principal minor of some matrix in $Q(A)$ is positive, and some $k \times k$ principal minor of some matrix in $Q(A)$ is negative. This proves the result. ■

The converse of Theorem 2.2 does not hold. For example, it is easy to check that the pattern

$$A = \begin{bmatrix} - & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$$

satisfies the minor conditions of Theorem 2.2, but A is not an IAP (see Chapter 5).

Proposition 2.3. If A is an IAP of order $n \geq 2$, then A has at least one positive diagonal entry, and A has at least one negative diagonal entry.

Proof. Since A is an IAP, there is $B_1 \in Q(A)$ such that

$$i(B_1) = (0, 1, n-1).$$

The one eigenvalue with negative real part must be negative, and the sum of the $n-1$ eigenvalues with real part equal to zero is zero. So, $\mathrm{tr}(B_1) = \sum \lambda_i < 0$. Hence, we have at least one negative diagonal entry. Similarly, with the use of $i(B_2) = (1, 0, n-1)$, we see that A has at least one positive diagonal entry. ■

Proposition 2.4. If A is an $n \times n$ $(n \geq 2)$ IAP, then A is not SNS, and A has two cycles of length n producing oppositely signed terms in det A.

Proof. Since A is an IAP, there is a $B_1 \in Q(A)$ such that

$$i(B_1) = (n, 0, 0).$$

The non-real eigenvalues occur in conjugate pairs. Hence, the product of the n eigenvalues with positive real part is positive. Thus, det $(B_1) > 0$. Since A is an IAP, we also have a $B_2 \in Q(A)$ such that

$$i(B_2) = (n-1, 1, 0).$$

The product of the $n-1$ eigenvalues with positive real part is positive, while the one eigenvalue with negative real part must be a negative real number. Hence, the product of the n eigenvalues is negative, so that det $(B_2) < 0$. Thus, the conclusions follow. ■

Proposition 2.5. If A is an IAP order 3, then $Q(A)$ allows a positive and a negative principal minor of order k for all $k = 1, 2, 3$.

Proof. We have seen that in general for an $n \times n$ IAP, the result is true for $k = 1$ and $k = n$ (by the two preceding propositions). Suppose $k = 2$. Now,

$$E_2(B) = \sum_{1 \leq i < j \leq 3} \lambda_i \lambda_j.$$

Let $B_1 \in Q(A)$ such that $i(B_1) = (3, 0, 0)$. If all 3 eigenvalues are positive, then $E_2(B_1) > 0$. If we have eigenvalues $a + bi$, $a - bi$, $c > 0$, $(a > 0)$, then $E_2(B_1) = (a^2 + b^2) + (ca + cbi) + (ca - cbi) = a^2 + b^2 + 2ca > 0$. Since $E_2(B_1) > 0$, some 2×2 principal minor of B_1 is positive.

Similarly, we have $B_2 \in Q(A)$ such that $i(B_2) = (1, 1, 1)$. So, B_2 has a positive eigenvalue λ_1, a negative eigenvalue λ_2 and a zero eigenvalue. Hence, $E_2(B_2) = \lambda_1 \lambda_2 < 0$, and some 2×2 principal minor of B_2 is negative. ■

Proposition 2.6. If A is an IAP order 4, then $Q(A)$ allows a positive and a negative principal minor of order k for all $k = 1, 2, 3, 4$.

Proof. From Proposition 2.3 and Proposition 2.4 , it is obvious that for $k = 1$ and $k = 4$ the conclusion is true. For $k = 2$, using the inertia triple $(4, 0, 0)$, we see that all four eigenvalues are positive, or have the form $a + bi$, $a - bi$, c,d $(a, c, d > 0)$, or $a + bi$, $a - bi$, $c + di$, $c - di$ $(a, c > 0)$.

For $a + bi$, $a - bi$, c, d,

$$E_2 = a^2 + b^2 + ca + cbi + ca - cbi + da + bi + da - bi + cd > 0.$$

Also, for $a + bi$, $a - bi$, $c + di$, $c - di$,

$$\begin{aligned} E_2 &= a^2 + b^2 + c^2 + d^2 + ac + adi + bci - bd + ac - adi + bci + \\ & \quad bd + ac - bci + adi + bd + ac - bci - adi - bd \\ &= a^2 + b^2 + c^2 + d^2 + 4ac > 0. \end{aligned}$$

We know that an IAP has a positive entry a_{ii} and a negative entry a_{jj}. So, by emphasizing these diagonal entries, we can get a minor $\det(B[i, j]) < 0$ for some $B \in Q(A)$ (as well as $E_2(B) < 0$).

For $k = 3$, using inertia triple $(3, 0, 1)$, we get $E_3 > 0$, and using inertia triple $(0, 3, 1)$ we get $E_3 < 0$. Thus, the conclusion follows. ■

A natural general question is then the following:

Question. If A is an IAP of order n, does $Q(A)$ allow a positive and a negative principal minor of order k for all $k = 1, 2, 3, \ldots, n$?

From Propositions 2.3 and 2.4, this is certainly the case for $k = 1$ and $k = n$. We now establish further general results to partially answer the above question.

Theorem 2.7 Let A be an IAP of order $n \geq 2$. Then A allows a positive and a negative principal minor of order 2.

Proof. By Proposition 2.3, A has a positive diagonal entry a_{jj} and a negative diagonal entry a_{kk}. By emphasizing these diagonal entries, we can get a matrix $B \in Q(A)$ with $E_2(B) < 0$. Thus, B has a negative principal minor of order 2.

We now show that A also allows a positive principal minor of order 2.

Let $B \in Q(A)$ with $i(B) = (2, 0, n-2)$. Write $\sigma(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}$ are pure imaginary (possibly 0), while λ_{n-1} and λ_n have positive real parts. Since the nonzero pure imaginary eigenvalues occur in conjugate pairs, we have

$$\sum_{k=1}^{n-2} \lambda_k = 0. \tag{1}$$

Squaring both sides of the above equation (1) yields

$$\left(\sum_{k-1}^{n-2} \lambda_k^2\right) + 2\left(\sum_{1 \leq i_1 < i_2 \leq n-2} \lambda_{i_1}\lambda_{i_2}\right) = 0. \tag{2}$$

Since each λ_k $(1 \leq k \leq n-2)$ is pure imaginary (possibly 0), we have

$$\sum_{k-1}^{n-2} \lambda_k^2 \leq 0.$$

It then follows from (2) that

$$\sum_{1\le i_1<i_2\le n-2} \lambda_{i_1}\lambda_{i_2} \ge 0. \tag{3}$$

Consider the sum

$$\sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2}. \tag{4}$$

The part of (4) that involves only the eigenvalues $\lambda_1,\ldots,\lambda_{n-2}$ is nonnegative by (3).

The part of (4) that involves exactly one of λ_{n-1} and λ_n is

$$\left(\sum_{k-1}^{n-2} \lambda_k\right)(\lambda_{n-1}+\lambda_n) = 0 \tag{5}$$

in view of (1).

The part of (4) involving both λ_{n-1} and λ_n is

$$\lambda_{n-1}\lambda_n > 0, \tag{6}$$

since λ_{n-1} and λ_n are either two positive numbers or a conjugate pair with positive real part.

Combining (3), (5) and (6), we see that

$$E_2(B) = S_2(\lambda_1,\lambda_2,\ldots,\lambda_n) = \sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} > 0.$$

It follows that B has a positive principal minor of order 2. ■

Theorem 2.8. Let A be an IAP of order $n \ge 3$. Then A allows a positive and a negative principal minor of order 3.

Proof. Let $B \in Q(A)$ with $i(B) = (3, 0, n-3)$. Write $\sigma(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n-3}$ are pure imaginary (possibly 0), while $\lambda_{n-2}, \lambda_{n-1}$ and λ_n have positive real parts. Since the nonzero pure imaginary eigenvalues occur in conjugate pairs, we have

$$\sum_{k=1}^{n-3} \lambda_k = 0. \tag{7}$$

Squaring both sides of the above equation yields

$$\left(\sum_{k-1}^{n-3} \lambda_k^2\right) + 2\left(\sum_{1\le i_1<i_2\le n-3} \lambda_{i_1}\lambda_{i_2}\right) = 0. \tag{8}$$

Since each λ_k $(1 \le k \le n-3)$ is pure imaginary (possibly 0), we have

$$\sum_{k-1}^{n-3} \lambda_k^2 \le 0.$$

It then follows from (8) that

$$\sum_{1\le i_1<i_2\le n-3} \lambda_{i_1}\lambda_{i_2} \ge 0 \tag{9}$$

Further, note that

$$\sum_{1\le i_1<i_2<i_3\le n-3} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} = 0, \tag{10}$$

since the terms come in pure imaginary conjugate pairs.

We now consider the sum

$$\sum_{1\le i_1<i_2<i_3\le n} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3}. \tag{11}$$

The part of (11) involving only eigenvalues in $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-3}\}$ is 0 by (10).

The part of (11) involving exactly one eigenvalue in $\{\lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$ is

$$\left(\sum_{1\le i_1<i_2\le n-3} \lambda_{i_1}\lambda_{i_2}\right)(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) \ge 0 \tag{12}$$

in view of (9) and $\lambda_{n-2} + \lambda_{n-1} + \lambda_n > 0$.

The part of (11) involving exactly two eigenvalues in $\{\lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$ is

$$\left(\sum_{k=1}^{n-3} \lambda_k = 0\right)\left(\sum_{n-2\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2}\right) = 0 \tag{13}$$

by (7).

The only term of (11) involving all three eigenvalues in $\{\lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$ is

$$\lambda_{n-2}\lambda_{n-1}\lambda_n > 0 \tag{14}$$

Combining (10) with (12)–(14), we see that

$$E_3(B) = S_3(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \le i_1 < i_2 < i_3 \le n} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} > 0. \tag{15}$$

It follows that B has a positive principal minor of order 3.

By considering a matrix $C \in Q(A)$ with $i(C) = (0, 3, n-3)$ instead, a similar proof as above shows that C has a negative principal minor of order 3. Alternatively, observe that $-A$ is also an IAP. From the above we know that $-A$ allows a positive principal minor of order 3. It then follows that A allows a negative principal minor of order 3. ■

Theorem 2.9. Let A be an IAP of order $n \ge 2$. Then A allows a positive and a negative principal minor of order $n-1$.

Proof. Let $B \in Q(A)$ with $i(B) = (n-1, 0, 1)$. Since the only eigenvalue with real part 0 must be 0 itself (denoted λ_n), and 0 does not make any contribution to the symmetric sum $S_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have

$$E_{n-1}(B) = S_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 \lambda_2 \ldots \lambda_{n-1} > 0.$$

Similarly, if $B \in Q(A)$ satisfies $i(B) = (n-2, 1, 1)$, then

$$E_{n-1}(B) = S_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 \lambda_2 \ldots \lambda_{n-1} < 0. \quad ■$$

Theorem 2.10. Let A be an IAP of order $n \ge 3$. Then A allows a positive principal minor of order $n-2$. If n is odd, then A allows a negative principal minor of order $n-2$

Proof. Suppose that $B \in Q(A)$ satisfies $i(B) = (n-1, 0, 1)$. Write $\sigma(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_n = 0$ is the only eigenvalue with real part 0. Since all nonzero eigenvalues of B have positive real parts and the nonreal eigenvalues come in conjugate pairs, we have

$$\lambda_1\lambda_2\ldots\lambda_{n-1} > 0. \tag{16}$$

Note that the reciprocals of the nonzero eigenvalues also have positive real parts and the nonreal ones come in conjugate pairs. Hence,

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{n-1}} > 0. \tag{17}$$

It follows that

$$\begin{aligned} E_{n-2}(B) \quad &= S_{n-2}(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &= \lambda_1\lambda_2\ldots\lambda_{n-1}\left(\tfrac{1}{\lambda_1} + \tfrac{1}{\lambda_2} + \cdots + \tfrac{1}{\lambda_{n-1}}\right) > 0. \end{aligned} \tag{18}$$

Therefore, A allows a positive principal minor of order $n-2$.

Suppose that n is odd. By considering a matrix $C \in Q(A)$ with $i(C) = (0, n-1, 1)$ instead, a similar argument as above shows that A allows a negative principal minor of order $n-2$, since the product of all the nonzero eigenvalues, $\lambda_1\lambda_2\ldots\lambda_{n-1}$, is still positive while the sum of the reciprocals of the nonzero eigenvalues is negative.

■

We conclude this section with a fact on 2×2 sign patterns.

Proposition 2.11. Up to equivalence,

$$T_2 = \begin{bmatrix} - & + \\ - & + \end{bmatrix}$$

is the only 2×2 SAP, IAP, MSAP and MIAP.

Proof. For

$$B = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix} \in Q(T_2),$$

$p_B(x) = x^2 - (d-a)x + bc - ad$, which yields every monic polynomial of degree 2 with real coefficients as a, b, c, d vary all positive numbers. So, T_2 is an SAP, hence also an IAP. Now, it is easily seen that any 2×2 IAP cannot have any zero entries.

So, then T_2 is a MSAP as well as a MIAP. By examination of all 2×2 entry-wise nonzero patterns, we can see that the following equivalent patterns are the only 2×2 IAP's:

$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & - \\ + & + \end{bmatrix}, \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \begin{bmatrix} + & - \\ + & - \end{bmatrix}.$$

Hence, we have all the needed results.

■

3. TREE SIGN PATTERNS

For a sign pattern matrix whose undirected graph is a tree, it is a fact that such a matrix is irreducible if and only if it is combinatorially symmetric , i.e., $a_{ij} \neq 0$ whenever $a_{ji} \neq 0$. Recall that we call such an irreducible sign pattern matrix a tree sign pattern matrix (tsp) matrix. Suppose A is an $n \times n$ tsp. Since $G(A)$ is a tree, $G(A)$ has $n-1$ edges. So, A has $2(n-1)$ off-diagonal nonzero entries. In addition, suppose A is an IAP. Then, we have a positive and a negative diagonal entry. Hence, A has at least $2n$ nonzero entries.

From Proposition 2.7, we see that T_2 is the only 2×2 tsp IAP.

Proposition 3.1. For $n = 3$, if A is a tsp, then A is permutation similar to a tridiagonal pattern.

Proof: Since $G(A)$ is a tree, we have nonzero entries such as $a_{i_1 i_2}$ and $a_{i_2 i_3}$. With a permutation similarity, we can assume a_{12}, a_{23} are nonzero. Hence,

$$A = \begin{bmatrix} * & a_{12} & * \\ a_{21} & * & a_{23} \\ * & a_{32} & * \end{bmatrix}.$$

Suppose $a_{13} \neq 0$ and therefore $a_{31} \neq 0$. Then a_{12}, a_{23}, a_{31} are nonzero and we have a cycle of length 3 in $G(A)$. However, $G(A)$ is a tree, so that we have a contradiction. Therefore, A is permutation similar to a tridiagonal pattern such as

$$\begin{bmatrix} * & a_{12} & 0 \\ a_{21} & * & a_{23} \\ 0 & a_{32} & * \end{bmatrix}.$$

■

In addition to A being a tsp, suppose A is an IAP, so that we have at least 2 nonzero diagonal entries which are positive and negative, respectively.

Could

$$A = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}?$$

No, since we have only one cycle of length 3 in A, by Proposition 2.4, A is not an IAP.

Next,

$$\begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, \quad \begin{bmatrix} + & + & 0 \\ + & 0 & + \\ 0 & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & + & + \end{bmatrix}, \quad \begin{bmatrix} + & + & 0 \\ - & 0 & + \\ 0 & + & - \end{bmatrix}$$

are each SNS and hence not IAP. For

$$\begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, \quad \begin{bmatrix} - & - & 0 \\ - & 0 & - \\ 0 & - & + \end{bmatrix}$$

if $B \in Q(A)$, any 2×2 minor of B is negative. Hence, by Proposition 2.5., A is not an IAP.

We are led to the following result [DJO00].

Proposition 3.2. Up to equivalence,

$$T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}$$

is the only 3×3 tsp MSAP (MIAP).

What about 3×3 tsp IAP's with all three of the diagonal entries are nonzero? From [DJO00], up to equivalence,

$$T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, \quad U = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}, \quad \tilde{T}_3 = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & - & + \end{bmatrix}$$

are the only 3×3 tsp SAP's (IAP's).

Generalizing T_3, we have the $n \times n$ antipodal pattern.

Theorem 3.3. For $2 \leq n \leq 16$,

$$T_n = \begin{bmatrix} - & + & 0 & \cdots & \cdots & 0 \\ - & 0 & + & \ddots & & \vdots \\ 0 & - & 0 & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & - & 0 & + \\ 0 & \cdots & \cdots & 0 & - & + \end{bmatrix}$$

is an SAP.

This is a theorem that follows from the results in [DJO00] and [EOD03]. T_n is in fact a MSAP (MIAP) since it has the minimal number ($2n$) of nonzero entries.

Up to equivalence, a 4×4 tsp A is a star pattern or a tridiagonal pattern. First, consider the 4×4 tsp SAP's (IAP's) with 8 nonzero entries. Up to equivalence, we have just

$$T_4 = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & + \end{bmatrix},$$

and

$$H = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}.$$

These patterns are actually MIAP's since they have the minimal number (8) of nonzero entries.

What about the 4×4 tsp IAP's with more than 8 nonzero entries? For $n \times n$ star patterns we have the following result from [MTD04].

Proposition 3.4. If $n \geq 2$ and S_n is a star sign pattern, then the following are equivalent:

1. S_n is equivalent to one of Y_n, Z_{np}, Z^+_{np} or Z^-_{np} (for appropriate p).

2. S_n is spectrally arbitrary.

3. S_n is inertially arbitrary.

4. S_n is potentially nilpotent and has a loop at each of the $n-1$ leaves in its graph.

5. S_n and -S_n are both potentially stable and S_n has a loop at each of the $n-1$ leaves in its graph.

6. S_n is potentially nilpotent and potentially stable. ■

We can then characterize the 4×4 SAP (IAP) star sign patterns.

Corollary 3.5. S_4 is an SAP (IAP) star sign pattern if and only if S_4 is equivalent to one of the following patterns:

$$Y_4 = \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}, \quad Z_{41} = \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & + \end{bmatrix}, \quad Z_{42} = \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix},$$

$$Z_{41}^{+} = \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & + \end{bmatrix}, \quad Z_{41}^{-} = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & + \end{bmatrix}, \quad Z_{42}^{+} = \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix},$$

$$Z_{42}^{-} = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix}.$$

We next analyze the 4×4 tridiagonal patterns.

4. 4×4 TRIDIAGONAL SPECTRALLY ARBITRARY SIGN PATTERNS

In the article [JS89], the authors identify the 4×4 potentially stable tridiagonal patterns up to equivalence. Since an IAP is a potentially stable pattern, to investigate the 4×4 tridiagonal IAP's we can consider these potentially stable tridiagonal patterns.

The following patterns from [JS89] are permutationally similar to a superpattern of T_4 and hence are SAP:

$$\begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}.$$

The following patterns from [JS89] are permutationally similar to a superpattern of H and hence are SAP:

$$\begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}.$$

For each pattern A in the following list, A^4 is not compatible with the 4×4 zero matrix. Therefore, these patterns are not SAP's.

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}.$$

Also, we have the following patterns which are not IAP's, since none of these patterns have both a positive and a negative diagonal entry.

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}.$$

The class of $n \times n$ potentially stable patterns is not closed under negation. If A is potentially stable, then $(0, n, 0) \in i(A)$, and hence $(n, 0, 0) \in i(-A)$. However, the class of $n \times n$ inertially arbitrary patterns (spectrally arbitrary patterns) is closed under negation. For example,

$$\begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}$$

is the negative of

$$\begin{bmatrix} + & - & 0 & 0 \\ + & + & - & 0 \\ 0 & + & - & - \\ 0 & 0 & + & + \end{bmatrix},$$

which is the transpose of

$$\begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}.$$

The first and third patterns are in the list of [JS89]. Hence, it suffices to analyze either the first or the third pattern. Similarly, we only need to analyze one of the two patterns in each of the following pairs:

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & + \end{bmatrix}$$

The Nilpotent-Jacobian method for showing that a pattern is SAP was originally given in [DJO00]. We will use a reformulation of this method which was given in [BMOD04].

Theorem 4.1. Let A be an $n \times n$ sign pattern, and suppose that there exists a nilpotent realization $M = [m_{ij}]$ of A with at least n nonzero entries, say, $m_{i_1 j_1}, \ldots, m_{i_n j_n}$. Let X be the matrix obtained by replacing these entries in M by variables $x_1, \ldots, x_n$, and let $(-1)^k \alpha_k$ be the coefficient of x^{n-k} in $p_X(x)$ for $k = 1, 2, , \ldots, n$. If the Jacobian $\frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(x_1, \ldots, x_n)}$ is nonzero at $(x_1, \ldots, x_n) = (m_{i_1 j_1}, \ldots, m_{i_n j_n})$, then A, as well as every superpattern of A, is spectrally arbitrary.

Using the Nilpotent-Jacobian method and MAPLE, we found more of the tridiagonal patterns which are SAP. In the following list we give the pattern, a nilpotent matrix in the corresponding sign pattern class, and the entries where we assign variables.

$$\begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 0 & -8 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ 0 & -c & -1 & 1 \\ 0 & 0 & 1 & -d \end{bmatrix};$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/2 & 1 & 1 & 0 \\ 0 & -1/2 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a & 1 & 1 & 0 \\ 0 & -b & -c & 1 \\ 0 & 0 & -d & 1 \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1/2 \\ 0 & 0 & -2 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & a & 0 & 0 \\ -1 & 0 & b & 0 \\ 0 & -1 & c & d \\ 0 & 0 & -1 & -1 \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}, \quad \begin{bmatrix} -1 & 1/3 & 0 & 0 \\ -1 & 0 & 1/3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & a & 0 & 0 \\ -1 & 0 & b & 0 \\ 0 & 1 & c & d \\ 0 & 0 & 1 & -d \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 & 0 & 0 \\ -1 & 2 & 1/2 & 0 \\ 0 & -1 & -1 & 1/2 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & a & 0 & 0 \\ -1 & b & c & 0 \\ 0 & -1 & -d & 1/2 \\ 0 & 0 & -1 & 0 \end{bmatrix};$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 & 0 \\ -1 & 0 & c & 0 \\ 0 & -1 & -2 & d \\ 0 & 0 & -1 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \quad \begin{bmatrix} 0 & 3/2 & 0 & 0 \\ -1 & -1 & 1/2 & 0 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & 0 & 0 \\ -1 & -b & 1/2 & 0 \\ 0 & -1 & -c & 6 \\ 0 & 0 & -1 & d \end{bmatrix};$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \quad \begin{bmatrix} 28/5 & 2401/100 & 0 & 0 \\ -1 & -23/5 & 5/4 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 & 0 \\ -1 & c & 5/4 & 0 \\ 0 & 1 & 0 & d \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

We illustrate the details of applying the N-J method with the following specific example.

Example 4.2. We want to show that the following sign pattern A is an SAP by using the Nilpotent-Jacobian method

$$A = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}.$$

Multiplying by a suitable scalar matrix and then performing a suitable diagonal similarity if necessary, we may assume that the $(1,1)$ entry and the first subdiagonal entries of $B \in Q(A)$ all have absolute value 1.

By using MAPLE,

$>$ with(linalg):

$>$ B:= matrix(4,4,[-1,a,0, 0, -1, b,c,0, 0, -1, -d ,e , 0, 0 ,-1, 0]);

$$B := \begin{bmatrix} -1 & a & 0 & 0 \\ -1 & b & c & 0 \\ 0 & -1 & -d & e \\ 0 & 0 & -1 & \end{bmatrix}$$

$> solve(trace(B), trace(B^2), trace(B^3), det(B), a, b, c, d, e);$

One set of solutions given by Maple is

$$\left\{d = -1 + b, e = \frac{1 - 2b + b^2}{b}, b = b, c = \frac{-1 + 3b - 3b^2 + b^3}{b}, a = b\right\}$$

By assigning $b = 2$, we get a nilpotent matrix N in $Q(A)$.

$> \;\; N := matrix(4, 4, [-1, 2, 0, 0, -1, 2, 1/2, 0, 0, -1, -1, 1/2, 0, 0, -1, 0]);$

$$N = \begin{bmatrix} -1 & 2 & 0 & 0 \\ -1 & 2 & 1/2 & 0 \\ 0 & -1 & -1 & 1/2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$> evalm(N^4);$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By considering 4 nonzero entries of N as variables, we get

$> C := matrix(4, 4, [-1, a, 0, 0, -1, b, c, 0, 0, -1, -d, 1/2, 0, 0, -1, 0]);$

$$\begin{bmatrix} -1 & a & 0 & 0 \\ -1 & b & c & 0 \\ 0 & -1 & -d & 1/2 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

> collect(charpoly$(C, x), x$);

$x^4+(d+1-b)x^3+(c-bd+a-b+1/2+d)x^2+((-1/2)b+ad+c+1/2-bd)x-(1/2)b+(1/2)a$

> Jacob := jacobian($[d+1-b, c-bd+a-b+1/2+d, (-1/2)b+ad+c+1/2-bd, -(1/2)b+(1/2)a], [a, b, c, d]$);

$$\text{Jacob} := \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1-d & 1 & -b+1 \\ d & (-1/2)-d & 1 & a-b \\ 1/2 & -1/2 & 0 & 0 \end{bmatrix}$$

> det(Jacob);

$$-3/4+d/2+a/2$$

> subs($\{a=2, d=1/2\}$, det(Jacob));

$$1/2$$

Since the Jacobian (1/2) is nonzero, the sign pattern A (and hence every superpattern of it) is an SAP. ■

5. CONNECTIONS WITH POTENTIALLY NILPOTENT PATTERNS

As we mentioned in the introduction, if a pattern A is spectrally arbitrary, then A is potentially nilpotent (PN). The authors in [BMOD04] have characterized the SAPs of order 3; they demonstrate that an irreducible 3×3 pattern is an SAP if and only if it is potentially nilpotent and has at least one positive and one negative entry on its main diagonal. Let

$$\mathcal{D}_{3,3} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & 0 & + \end{bmatrix}, \mathcal{U} = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}, \mathcal{D}_{3,2} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, \mathcal{V} = \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ - & + & + \end{bmatrix}.$$

Of course, $\mathcal{D}_{3,2}$ is the same as T_3, and the following result is proved in [CV05].

Theorem 5.1. If A is a pattern of order 3, then the following statements are equivalent:

1) A is spectrally arbitrary.
2) A is inertially arbitrary.
3) Up to equivalence, A is a superpattern of $\mathcal{D}_{3,3}$, $\mathcal{U}$, $\mathcal{D}_{3,2}$, or $\mathcal{V}$.

In fact, [BMOD04] shows that the 3×3 minimal spectrally arbitrary sign patterns are precisely the patterns that are equivalent to one of the patterns $\mathcal{D}_{3,3}$, $\mathcal{U}$, $\mathcal{D}_{3,2}$, or $\mathcal{V}$. We also see from Theorem 5.1 that all the 3×3 inertially arbitrary patterns are irreducible (the directed graphs are strongly connected).

All of this leaves the following question: Can we have a 3×3 irreducible potentially nilpotent pattern that is not spectrally arbitrary? The answer is in fact yes, and provided by the list of patterns from [GLS07]. The irreducible 3×3 potentially nilpotent patterns (up to equivalence) include

$$N = \begin{bmatrix} 0 & + & + \\ - & 0 & - \\ + & - & 0 \end{bmatrix}$$

which is not even IAP because of the zero diagonal. (Note that $\begin{bmatrix} 0 & + & + \\ 0 & 0 & + \\ 0 & 0 & 0 \end{bmatrix}$ is a reducible potentially nilpotent sign pattern, but not an IAP.)

In fact, up to equivalence, the above sign pattern N is the only irreducible 3×3 potentially nilpotent pattern that does not have a positive and a negative diagonal entry. Hence, from the above results, up the equivalence, this pattern is the only irreducible 3×3 potentially nilpotent pattern that is not an IAP.

In [BMOD04], an irreducible 4×4 potentially nilpotent pattern that is not an IAP is given, namely

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & - & 0 & + \\ - & 0 & 0 & - \end{bmatrix}$$

(Note that this pattern is not a tsp.)

In [CV05], an irreducible 4×4 IAP that is potentially nilpotent but not an SAP is given, namely

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & - & - \\ + & + & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix}$$

Also, in [CV05], a 7×7 reducible IAP that is not PN is given that is a direct sum of an irreducible 2×2 IAP and an irreducible 5×5 pattern that is not an IAP. However, this does not imply the existence of a family of irreducible $n \times n$ patterns that are IAP but not PN.

In [KOD07], for odd $n \geq 5$, a new family of irreducible IAP's which are not potentially nilpotent (and so are not SAP's) is presented.

The classes of patterns are related in a very interesting Venn diagram. We have the proper set inclusions even amongst irreducible patterns.

In order to present a powerful technique to prove that a certain sign pattern does not allow nilpotence, we introduce the following concepts. Consider a subset

S of the multivariable polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. A *zero* or a *solution* of S in $\mathbb{R}$ is an n-tuple $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ with $p(r_1, r_2, \ldots, r_n) = 0$ for every polynomial $p \in S$. It can be seen that an n-tuple $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ is a solution of S iff it is a solution of the ideal generated by S. Hilbert's Basis Theorem states that every ideal of a polynomial ring over a field is finitely generated. Let M be a set of monomials in $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Suppose certain ordering of all the monomials is prescribed. Let $LT(P)$, the leading term of a polynomial P, be the largest monomial appearing in P. If S be a subset of $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and $LT(S)$ is the ideal generated by $\{LT(s) : s \in S\}$. Let I be an ideal of $\mathbb{R}[x_1, x_2, \ldots, x_n]$. A finite subset $G = \{g_1, g_2, \ldots, g_k\}$ of I is called a *Gröbner basis* of I if $\{LT(g_1), LT(g_2), \ldots, LT(g_k)\}$ generates $LT(I)$. For any ideal I of $\mathbb{R}[x_1, x_2, \ldots, x_n]$ the following are true.

(1) I has a Gröbner basis relative to any monomial ordering.

(2) Every Gröbner basis G of I generates I.

It can be seen that for every subset S of $R[x_1, x_2, \ldots, x_n]$ and a Gröbner basis G of the ideal generated by S, the solution set of S is the same as the solution set of G.

It can be shown that an $n \times n$ real matrix B is nilpotent iff $\operatorname{tr}(B) = 0$, $\operatorname{tr}(B^2) = 0$, $\operatorname{tr}(B^3) = 0, \ldots,$ $\operatorname{tr}(B^n) = 0$. This result remains valid when the last condition $\operatorname{tr}(B^n) = 0$ is replaced by $\det(B) = 0$.

An SAP must be potentially nilpotent. Using Maple to compute a Gröbner basis of polynomials obtained using the necessary and sufficient conditions mentioned in the above remark for a matrix to be nilpotent, we can show that certain sign patterns do not allow nilpotence, as the following example shows.

Example 5.2. The following sign pattern is not potentially nilpotent (and hence is not an SAP):

$$A = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix}.$$

Proof. By performing a suitable diagonal similarity if necessary, we may assume that a matrix $B \in Q(A)$ has the following form

$$B = \begin{bmatrix} -a & b & 0 & 0 \\ -1 & c & d & 0 \\ 0 & -1 & -e & f \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where the variables can take on any positive values. Using $\mathrm{tr}(B) = 0$, $\mathrm{tr}(B^2) = 0$, $\mathrm{tr}(B^3) = 0$, $\det(B) = 0$, we get the following system of polynomial equations.

$$\begin{cases} a + c - e = 0 \\ a^2 - 2b + c^2 - 2d + e^2 + 2f = 0 \\ -a^3 + 3ab - 3bc + c^3 - 3cd + 3de - e^3 - 3ef = 0 \\ acf - bf = 0 \end{cases}$$

Within Maple, using the command

with(Groebner): gbasis($\{\mathrm{trace}(B), \mathrm{trace}(B^2), \mathrm{trace}(B^3), \det(B)\}, \mathrm{tdeg}(a, b, c, d, e, f)$);

we get the Gröbner basis G with respect to the total degree ordering (a, b, c, d, e, f) for the set of the polynomials in the above system of equations

$$G = \{a - c + e, c^2 - ce + e^2 - b - d + f, -df + e^2 f + f^2, fcd - dfe + ef^2,$$
$$e^3 + cd - 2de + 2ef, \mathbf{bdf} + \mathbf{cef^2}, bdef + ef^2 d - cf^3 - ef^3, bd^2 f + d^2 f^2 - 2df^3 + f^4\}$$

Note that the solutions of the original system of equations are the same as the zeros of the system of polynomials in a Gröbner basis. The equation $bdf + cef^2 = 0$ can not have a positive solution for a, b, c, d, e, f. Thus the original system can not have a solution where all the variables are positive. That is to say, A is not potentially nilpotent. ■

6. SOME OPEN QUESTIONS

In this chapter, we summarize the open questions that we have mentioned previously and also give some other interesting questions.

If A is an IAP pattern, is it necessary that every irreducible component of A is IAP?

We have seen that if A is an irreducible 3×3 pattern (with a positive and negative diagonal entry), then A is an SAP if and only if A is PN. How far can we extend this result?

We saw by examples that it is not true in general for irreducible 4×4 patterns. But, what about for the class of 4×4 tree sign patterns? In this case, is PN (together with positive and negative diagonal entries) equivalent to being SAP? In this connection, recall proposition 3.4. For star patterns A is SAP if and only if A is PN and PS (Potentially Stable). So, can we find a 4×4 star pattern that is PN but not PS?

More generally, can we produce (up to equivalence) the irreducible 4×4 PN patterns (or, say the star or tridiagonal ones)?

For PN patterns, is rational realization always possible? That is, if A is PN, does there always exist a rational $B \in Q(A)$ that is nilpotent?

In general, if A is PN and has nonzero diagonal entries, can we always find a nonzero Jacobian?

In [CV07], the authors give (up to equivalence) the irreducible 4×4 MIAP's that are not an SAP, namely

$$\begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix}$$

Can this result be extended to 5×5 patterns, or $n \times n$, in general?

References

[BMQ68] L. Bassett, J.S. Maybee and J. Quirk, Qualitative economics and the scope of the correspondence principle, *Econometrica* 36:544-563, 1968.

[BMOD04] T. Britz, J.J. McDonald, D.D. Olesky,and P.van den Driessche,Minimal Spectrally Arbitrary patterns, *SIAM J.Matrix Anal. Appl.* 26:257-271, 2004.

[BS95] R.A. Brualdi and B.L. Shader, *Matrices of Sign-Solvable Linear Systems*, Cambridge University Press, Cambridge, 1995.

[CV05] M. S. Cavers,and K. N. Vander Meulen, Spectrally and inertially arbitrary sign patterns, *Linear Algebra Appl.* 394:53-72, 2005.

[CV07] M. S. Cavers,and K. N. Vander Meulen, Inertially Arbitrary Nonzero Patterns of order 4.

[DJO00] J.H. Drew, C.R. Johnson, D.D. Olesky and P. van den Driessche, Spectrally arbitrary patterns, *Linear Algebra Appl.* 308:121-137, 2000.

[EOD03] L. Elsner, D.D. Olesky, P. van den Driessche, Low rank perturbations and the spectrum of a tridiagonal sign pattern, *Linear Algebra Appl.* 374:219-230, 2003.

[Esc87] C.A. Eschenbach, Eigenvalue Classification in Qualitative Matrix Analysis, doctoral dissertation directed by C.R. Johnson, Clemson University, 1987.

[EL99] C.A. Eschenbach and Z. Li, Potentially nilpotent sign pattern matrices, *Linear Algebra Appl.* 299:81-99, 1999.

[GL01] Y. Gao, J. Li, On the potential stability of star sign pattern matrices, *Linear Algebra Appl.* 327:61-68, 2001.

[GLS07] Y. Gao, Z. Li,and Y. Shao, Sign patterns allowing nilpotence of index 3, *Linear Algebra Appl.*, in press.

[GS01] Y. Gao and Y. Shao, Inertially arbitrary patterns, *Linear and Multilinear Algebra* 49(2):161-168, 2001.

[HL07] F.J. Hall and Z. Li, Sign Pattern Matrices, Chapter 33 in *Handbook of Linear Algebra,* Chapman and Hall CRC Press, Boca Raton, 2007.

[HLW01] F.J. Hall, Z. Li and D. Wang, Symmetric Sign Pattern Matrices That Require Unique Inertia, *Linear Algebra Appl.* 338:153-169, 2001.

[JKD77] C. Jeffries, V. Klee and P. van den Driessche, When is a matrix sign stable?, *Canadian J. Math.* 29:315-326, 1977.

[JMO97] C.R. Johnson, J.S. Maybee, D. D. Olesky and P. van den Driessche, Nested sequences of principal minors and potential stability *Linear Algebra Appl.* 262:243-257, 1997.

[JS89] C.R. Johnson and T.A. Summers, The potentially stable tree sign patterns for dimensions less than five, *Linear Algebra Appl.* 126:1-13, 1989.

[KOD07] I.J.Kim, D.D. Olesky, P. van den Driessche. Inertially arbitrary sign patterns with no nilpotent realization. *Linear Algebra Appl.* 421:264-283, 2007.

[MOT03] J.J. McDonald, D.D. Olesky, M. Tsatsomeros and P. van den Driessche, On the spectra of striped sign patterns, *Linear and Multilinear Algebra* 51:39-48, 2003.

[ML02] Z. Miao and J. Li, Inertially arbitrary $(2r-1)$-diagonal sign patterns, *Linear Algebra Appl.* 357:133-141, 2002.

[MTD04] G. MacGillivray, R.M. Tifenbach, P. van den Driessche, Spectrally Arbitrary Star Sign Patterns, *Linear and Multilinear Algebra* , 10.1016, 2004.

[QR65] J. Quirk and R. Ruppert, Qualitative economics and the stability of equilibrium, *Rev. Economic Studies* 32:311-326, 1965.

[Sam47] P.A. Samuelson, *Foundations of Economic Analysis*, Harvard University Press, Cambridge, 1947, Atheneum, New York, 1971.

[SSG04] Y. Shao, L. Sun and Y. Gao, Inertia sets of two classes of symmetric sign patterns, *Linear Algebra Appl.* 384:85-95, 2004.

[Yeh96] L. Yeh, Sign patterns that allow a nilpotent matrix, *Bull. Austral. Math. Soc.* 53:189-196, 1996.